# Profit maximization with two goods and second-order conditions

A product is sold in two different markets where the price-output laws are respectively $p_1 = 26 - q_1$ and $p_2 = 40 - 4q_2$. The cost function is $C = q_1^2 + 2q_1q_2 + q_2^2$. Calculate:

a) The level of production that must be allocated to each market and the prices at which it must be sold to obtain the maximum profit.

b) The relationship between the selling price and the elasticity of demand.

c) Classify the good according to the demand laws of each market.

# Solution

1.
$$C = q_1^2 + 2q_1q_2 + q_2^2$$
$$p_1 = 26 - q_1$$
$$p_2 = 40 - 4q_2$$

We set up the profit function: $B = I - C = (p_1 \cdot q_1 + p_2 \cdot q_2) - (q_1^2 + 2q_1q_2 + q_2^2)$
We replace:
$$B = (26 - q_1) \cdot q_1 + (40 - 4q_2) \cdot q_2 - (q_1^2 + 2q_1q_2 + q_2^2)$$

We order:
$$B = 26q_1 - q_1^2 + 40q_2 - 4q_2^2 - q_1^2 - 2q_1q_2 - q_2^2$$

Finally, the function to optimize is:
$$B = 26q_1 - 2q_1^2 + 40q_2 - 5q_2^2 - 2q_1q_2$$

First-order conditions (FOC):
$$B'_{q_1} = 26 - 4q_1 - 2q_2 = 0$$
$$B'_{q_2} = 40 - 10q_2 - 2q_1 = 0$$

To isolate we use the substitution method, just like before:
$$\frac{26 - 4q_1}{2} = q_2$$
$$13 - 2q_1 = q_2$$

We replace in the other equation and isolate:
$$40 - 10 \cdot (13 - 2q_1) - 2q_1 = 0$$
$$40 - 130 + 20q_1 - 2q_1 = 0$$
$$18q_1 = 90$$
$$q_1 = 5$$

Therefore:
$$13 - 2 \cdot (5) = q_2$$
$$3 = q_2$$

Finally, the critical point is $P = (5; 3)$
Second-order conditions (SOC):
$$B''_{q_1q_1} = -4$$
$$B''_{q_1q_2} = -2 \text{ (By the Schwarz Theorem)}$$
$$B''_{q_2q_2} = -10$$

We construct the Hessian:
$$\mathbf{H} = \begin{pmatrix} B''_{q_1q_1} & B''_{q_1q_2} \\ B''_{q_1q_2} & B''_{q_2q_2} \end{pmatrix}$$

$$\mathbf{H} = \begin{pmatrix} -4 & -2 \\ -2 & -10 \end{pmatrix} = [(-4)\cdot(-10)] - [(-2)\cdot(-2)]$$
$$\mathbf{H} = 40 - 4 = 36$$

$\mathbf{H} = 36 > 0$ we can confirm that there is a relative extremum.

To assess whether it is a maximum or minimum, we look at the sign of the second derivative $B''_{q_1 q_1}$:

$$B''_{q_1 q_1} = -4 < 0 \text{ there is a maximum at } P = (5;3) \text{ and the value of the total profit is:}$$

We replace in the objective function:

$$B = 26q_1 - 2q_1^2 + 40q_2 - 5q_2^2 - 2q_1 q_2$$
$$B = 26 \cdot 5 - 2 \cdot 5^2 + 40 \cdot 3 - 5 \cdot 3^2 - 2 \cdot 5 \cdot 3$$
$$B_{\text{total}} = 125$$

To obtain the prices, we replace in their respective demand functions:

$$p_1 = 26 - q_1 = 26 - 5 \Rightarrow p_1 = 21$$

$$p_2 = 40 - 4q_2 = 40 - 4 \cdot (3) \Rightarrow p_2 = 28$$

Answer: The level of production that must be allocated to market 1 is 5 units at $21each, andtomarket2is3unitsat28$ each. The total profit will be 125.

2. Relationship between selling price and demand elasticity.

$$E(p_1, q_1) = \frac{\partial q_1}{\partial p_1} \cdot \frac{p_1}{q_1} = -\frac{1}{5} \cdot \frac{21}{5} = -\frac{21}{25}$$

$$E(p_2, q_2) = \frac{\partial q_2}{\partial p_2} \cdot \frac{p_2}{q_2} = -\frac{1}{4} \cdot \frac{28}{3} = -\frac{7}{3}$$

The firm will decide to put a higher price on the good whose demand is less elastic. Since the demand for good 1 is more elastic than that of good 2, the price of good 2 will be higher than the price of good 1.

$$\frac{21}{25} > \frac{7}{3} \Rightarrow E_1 > E_2 \Rightarrow p_2 > p_1$$

3. In both markets, the good is typical since the derivatives of the demands with respect to the price are negative.